

Basket Option Pricing under Jump Diffusion Models

Ali Safdari-Vaighani

Abstract—Pricing financial contracts on several underlying assets received more and more interest as a demand for complex derivatives. The option pricing under asset price involving jump diffusion processes leads to the partial integral differential equation (PIDEs), which is an extension of the Black-Scholes PDE with a new integral term. The aim of this paper is to show how basket option prices in the jump diffusion models, mainly on the Merton model, can be computed using RBF based approximation methods. For a test problem, the RBF-PU method is applied for numerical solution of partial integral differential equation arising from the two-asset European vanilla put options. The numerical result shows the accuracy and efficiency of the presented method.

Keywords—Radial basis function, basket option, jump diffusion, RBF-PUM.

I. INTRODUCTION

THE Black-Scholes PDE and its extensions are the basic and the most well-known modeling for valuation of options with one underlying asset as well as basket options. The original Black-Scholes equation [1] is based on dynamics of asset prices with pure diffusion models. In most cases, pure diffusion models cannot interpret the empirical observations that comes from stock markets. Under actual market conditions, stock prices expose large and sudden changes when reacting to good or bad news. Jump-diffusion models extend the classical diffusion modeling framework by adding jumps to the diffusion dynamics. Merton [2] introduced the first jump diffusion process for modeling the behavior of stock prices.

For several underlying assets, the corresponding Black-Scholes model is a high-dimensional PDE equation, which needs to be solved by numerical methods. Pettersson et al. [3] present the RBFs for multi-dimensional European option and both one and two dimensional American options by Fasshauer et al. [4]. Recently, Safdari et al. [5] introduced a collocation partition of unity with local RBF approximations for American basket option pricing problem under Black-Scholes PDE. In many cases, however, an explicit closed-form valuation of options in jump diffusions is not possible and one is restricted to numerical procedures. Reference [6] developed a semi implicit approach for American options using a traditional linear complementarity solver for jump diffusion problems. Explicit time stepping for integral term is applied by [7]. An implicit, finite difference approach for single asset American and European options under the Merton jump diffusion model was explored in [8]-[10]. A RBF approximation method is applied for option pricing with single asset in exponential Levy models in [11].

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Two asset American claims under jump diffusion were priced using a Markov chain approach in [12]. Clift and Forsyth [13] introduced the finite difference method for numerical solution of two asset jump diffusion models for option valuation.

II. BASKET OPTION PRICING

To reproduce a more realistic behavior of the underlying assets, we assume that the asset price $S_i, i = 1 \dots d$ follows the risk-natural process

$$\frac{dS_i}{S_i} = (\mu - q_i)dt + \sigma_i dW_i + (e^{J_i} - 1)S_i dq, \quad (1)$$

where μ denotes a constant expected rate of return, σ_i and q_i are volatility and dividend of the underlying i th-asset, respectively. Here, W_i is the standard Brownian motion where ρ_{ij} is correlation between W_i, W_j . In (1) dq is Poisson process with the mean arrival rate $\lambda > 0$ and J_i is jump size of the i th-asset.

Using the Ito's formula for finite activity jump processes, the contingent claim $V(S, t)$ that depends on $S = (S_1, \dots, S_d) \in \tilde{\Omega} = \mathbb{R}_+^d$ can be derived by taking the expectation under the risk natural process. The resulting PIDE is given by

$$\begin{aligned} \frac{\partial V}{\partial t} = & -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} - \sum_{i=1}^d (r - q_i) S_i \frac{\partial V}{\partial S_i} \\ & + rV - \lambda \int_{\tilde{\Omega}} [V(S e^J, t) - V(S, t)] \\ & - \sum_{i=1}^d S_i (e^{J_i} - 1) \frac{\partial V}{\partial S_i}(S, t) g(J) dJ \quad (S, t) \in \tilde{\Omega} \times [0, T), \end{aligned} \quad (2)$$

where the jump magnitudes $J = (J_1, \dots, J_d)$ have some known probability density $g(J)$. In merton model, the density function for the jump magnitudes follow the normal distribution with mean vector $\tilde{\mu}$ and covariance matrix Σ as

$$g(J) = (2\pi)^{-d/2} (\det \Sigma)^{-1/2} \exp \left(-\frac{1}{2} (J - \tilde{\mu})^t \Sigma^{-1} (J - \tilde{\mu}) \right). \quad (3)$$

Note that we assume that there is a single Poisson process which derives correlated jumps in all assets. This corresponds to the single market stock process which affects both prices.

Let $S_i = e^{x_i}$ and $\tau = T - t$ and by the change of variables,

$V(e^x, T - \tau) = U(x, \tau)$ the equation becomes

$$\begin{aligned} \frac{\partial U}{\partial \tau} = & \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^d (r - q_i - \frac{\sigma_i^2}{2}) \frac{\partial U}{\partial x_i} \\ & - rU + \lambda \int_{\Omega} [U(x + J, \tau) - U(x, \tau) - \\ & \sum_{i=1}^d (e^{J_i} - 1) \frac{\partial U}{\partial x_i}(x, \tau)] g(J) dJ \quad (x, \tau) \in \Omega \times (0, T], \end{aligned} \quad (4)$$

By separating the integral part of equation and using the property of distribution function $g(J)$, $\int_{\Omega} g(J) dJ = 1$ the equation can be rewritten in the more tractable form:

$$\frac{\partial U}{\partial \tau}(x, \tau) = \mathcal{D}U(x, \tau) + \mathcal{I}U(x, \tau) \quad (x, \tau) \in \Omega \times (0, T], \quad (5)$$

where $\mathcal{D}U = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 U}{\partial x_i \partial x_j}(x, \tau) + \sum_{i=1}^d (r - q_i - \frac{\sigma_i^2}{2} - \kappa_i \lambda) \frac{\partial U}{\partial x_i}(x, \tau) - (r + \lambda)U(x, \tau)$ and $\mathcal{I}U = \lambda \int_{\Omega} U(x + J, \tau) g(J) dJ$

A. Terminal and Boundary Conditions

At the maturity T or $\tau = 0$ the typical option value simply equals its payoff value g , i.e.

$$U(x, 0) = g(x), \quad x \in \Omega. \quad (6)$$

The typical payoff value for basket put option is given by

$$g(x) = \max(E - \sum_{i=1}^d \alpha_i e^{x_i}, 0) \quad (7)$$

where E is the exercise price of the option and $\alpha_i, i = 1, \dots, d$ are given constants. The boundary of the computational domain can be divided into two parts: The near-field boundary, where one or more asset prices are zero, and the far-field boundary, where one or more asset-prices tend to infinity.

For the near-field boundary, it can be noted that once S_i reaches zero, it will be worthless afterwards, i.e., the solution remains at the boundary. We denote the d near-field boundaries by $\Omega_i = \{S \in \Omega | S \neq 0, S_i = 0\}$, $i = 1, \dots, d$. Then, the boundary values at Ω_i can be propagated by solving a $(d - 1)$ -dimensional PIDE problem. In this case, the payoff function of reduced basket put option is

$$g(x) = \max(E - \sum_{j=1, j \neq i}^d \alpha_j e^{x_j}, 0) \quad x \in \Omega_i, \quad i = 1, \dots, d. \quad (8)$$

We denote the solutions of the reduced problems by h_i and use the boundary conditions

$$U(x, t) = h_i(x, t), \quad x \in \Omega_i, \quad i = 1, \dots, d. \quad (9)$$

For put options, the contract becomes worthless as the price of any of the underlying assets tends to infinity. Therefore, we employ the following far-field boundary conditions:

$$\lim_{x_i \rightarrow \infty} U(x, t) = 0, \quad x \in \Omega, \quad i = 1, \dots, d. \quad (10)$$

B. Computational Domain

We have to restrict the domain $\Omega := \mathbb{R}^d$ of the space variable in the integral term to the bounded domain. First we take the linear transformation in integral term in (4)

$$\int_{\Omega} U(x + J, \tau) g(J) dJ = \int_{\Omega} U(J, \tau) g(J - x) dJ. \quad (11)$$

According to the asymptotic behavior of the price of option, there exists the compartment bounded domain $\Omega_J \subset \Omega$ such that we can divide the integral term (11) into two parts

$$\begin{aligned} \int_{\Omega} U(J, \tau) g(J - x) dJ = & \int_{\Omega_J} U(J, \tau) g(J - x) dJ \\ & + \int_{\Omega \setminus \Omega_J} U(J, \tau) g(J - x) dJ \end{aligned} \quad (12)$$

such that the value of the second integral part in (12) is less than a given tolerance. This procedure is described in [14]. The truncated domain is small in practice because the probability density function $g(J - y)$ goes to zero very quickly. We define $R(\tau, x, \Omega_J)$ by

$$R(\tau, x, \Omega_J) = \int_{\Omega \setminus \Omega_J} U(J, \tau) g(J - x) dJ. \quad (13)$$

As well, the residual $R(\tau, x, \Omega_J)$ is asymptotically zero in regions where the solution is asymptotically linear; linearity is a common assumption for far-field boundary conditions in finance [15]. We shall set the size of truncated domain Ω_J so that the $R(\tau, x, \Omega_J)$ is small enough. We have now replaced the integral on an infinite domain by finite one. Furthermore, it is possible to approximate the truncated integral by some kind of Newton-Cotes integration method which explained in next section.

For the numerical approximation propose, we can truncate the domain corresponding to the PDE part on (4). Assume that the computational bounded domain is Ω_X which $\Omega_J \subset \Omega_X$. For the numerical implementation, we can consider Ω_X as a general computational domain and set $\mathcal{I}U = 0, \forall J \in \Omega_X \setminus \Omega_J$ which cases to eliminate the integral term. Typically, the domain Ω_X is sufficiently large compared to the Ω_J which allows to control better the error from boundary approximations [13].

III. RADIAL BASIS FUNCTION COLLOCATION SCHEMES

For scalar function values f_j at scattered distinct node locations $x_j \in \mathbb{R}, j = 1, \dots, N$, the standard RBF interpolant takes the form

$$s(x) = \sum_{j=1}^N \lambda_j \phi(\|x - x_j\|), \quad (14)$$

where ϕ is a real-valued function such as the inverse multiquadric (IMQ) $\phi(r) = \frac{1}{\sqrt{\varepsilon^2 r^2 + 1}}$ coefficients $\lambda_j \in \mathbb{R}$ for $j = 1, \dots, N$, are determined by interpolation conditions $s(x_i) = f_i, i = 1, \dots, N$. In matrix form, the coefficient vector $\bar{\lambda} = [\lambda_1, \dots, \lambda_N]^T$ can be obtained by solving linear system

$$A \bar{\lambda} = f, \quad (15)$$

where $A_{ij} = \phi(\|x_i - x_j\|)$, $i, j = 1, \dots, N$, and $f = [f_1, \dots, f_N]^T$. If we define $\bar{\phi}(x) = (\phi(\|x - x_1\|), \dots, \phi(\|x - x_N\|))$, when $\bar{\lambda}$ is known, the RBF interpolant (14) can be rewritten as

$$s(x) = \bar{\phi}(x)\bar{\lambda} = \bar{\phi}(x)A^{-1}f. \quad (16)$$

We notice that matrix A is invertible for distinct node points where RBF function is positive definite such as IMQ.

For the approximation proposes, we need to apply the linear operator \mathcal{L} on (17) to evaluate the $s_{\mathcal{L}} = [\mathcal{L}s(x_1), \dots, \mathcal{L}s(x_N)]^T$ at the set of node points $X = \{x_i\}_{i=1}^N$. This leads to

$$s_{\mathcal{L}} = \Phi_{\mathcal{L}}A^{-1}f. \quad (17)$$

where $\Phi_{\mathcal{L}} = [\mathcal{L}\phi(\|x_i - x_j\|)]_{i,j=1,\dots,N}$. The differentiation matrix $D_{\mathcal{L}}$ under operator \mathcal{L} is given by $D_{\mathcal{L}} = \Phi_{\mathcal{L}}A^{-1}$.

Remark: The arising differentiation matrix is based on the standard global RBF interpolation method which is dense matrix. The differentiation matrices can be achieved based on the local properties of the RBF interpolation such as partition of unity RBF method in which the arising differentiation matrices are sparse and have the well performance for the high dimensional problems [16].

When we are dealing with the time-dependent PDE problem with the solution $f(x, t)$, the RBF approximation method takes the form

$$s(x, t) = \bar{\phi}(x)A^{-1}f(t), \quad (18)$$

where $f(t) = [f_1(t), \dots, f_N(t)]^T$, and $f_j(t) \approx f(x_j, t)$.

IV. RBF APPROXIMATION METHOD FOR MERTON OPTION PRICING MODEL

Using the RBF approximation – and the collocation PIDE (5) at the node points we get the linear system of ODEs

$$U'_I(\tau) = \mathcal{D}U_I(\tau) + \mathcal{I}U_I(\tau), \quad (19)$$

where $U_I(\tau) = [U_1(\tau), \dots, U_{N_I}(\tau)]^T$. The $\mathcal{D}U_I(\tau)$ and $\mathcal{I}U_I(\tau)$ correspond to the PDE part and Integral part, respectively.

$$\begin{aligned} \mathcal{D}U_I(\tau) = & \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j D_{ij,I} U_I(\tau) \\ & + \sum_{i=1}^d (r - q_i - \frac{\sigma_i^2}{2}) D_{i,I} U_I(\tau) - r U_I(\tau) + F(\tau) \end{aligned} \quad (20)$$

where $D_{.,I}$ contains the columns of the differentiation matrix corresponding to interior points, and

$$\begin{aligned} F(\tau) = & \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j D_{ij,b} F_b(\tau) \\ & + \sum_{i=1}^d (r - q_i - \frac{\sigma_i^2}{2}) D_{i,b} F_b(\tau) \end{aligned} \quad (21)$$

forces the boundary conditions, where $D_{.,b}$ contains the columns of the differentiation matrix corresponding to boundary points and $F_b(\tau) = [U(x_{N_I+1}, \tau), \dots, U(x_N, \tau)]^T$

contains the known boundary culmens. In simple form notation 20 can be rewritten as

$$\mathcal{D}U_I(\tau) = \mathcal{D}_{\mathcal{L}}U_I(\tau) + F(\tau) \quad (22)$$

where $\mathcal{D}_{\mathcal{L}} = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j D_{ij,I} + \sum_{i=1}^d (r - q_i - \frac{\sigma_i^2}{2}) D_{i,I} - rI$ and I is identity $N_I \times N_I$ matrix.

A. Discretization of Integral Term

For having the accurate approximation of the integral, we need the refinement in the integration region. A mapping is formed between a vector $X = (x_1, \dots, x_{N_I}) \in \Omega_X$ over the nodes of the collocation points in the vector $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_{N_r}) \in \Omega_J$ on the refined nodes. The mapping can be written as a $N_r \times N_I$ projection matrix R so that $U(\tilde{X}, \cdot) = RU(X, \cdot)$.

Let \tilde{X} be the vector with equidistance elements, and h be the same distance of the nodes in each direction. To approximate the integral term, we use the quadrature rule such as trapezoidal quadrature rule based on the refined node \tilde{X} . For example in the two dimensional case, (d=2), the discretization of the integral term on equation – in each collocation points x_i can be approximated as follow

$$\begin{aligned} \mathcal{I}U(x_i, \tau) = & \int_{\Omega_J} U(J, \tau) g(J - x_i) dJ \\ \approx & h^2 \sum_{j=1}^{N_r} w_j U(\tilde{x}_{1j}, \tilde{x}_{2j}, \tau) g(\tilde{x}_{1j} - x_{1i}, \tilde{x}_{2j} - x_{2i}) \end{aligned} \quad (23)$$

where w_j is 1 for points in the interior, 0.5 along the outer edges and 0.25 in the corner points of integration domain. In the matrix form the integral term can be written as

$$\mathcal{I}U_I(\tau) = W R U_I(\tau) \quad (24)$$

where the matrix W includes the trapezoid weights and density function g ,

$$W_{i,j} = [h^2 w_j g(\tilde{x}_{1j} - x_{1i}, \tilde{x}_{2j} - x_{2i})], i = 1, \dots, N_I, j = 1, \dots, N_r. \quad (25)$$

by replacing (22) together (24) in (19), the ODEs system will be rewritten as

$$U'_I(\tau) = (\mathcal{D}_{\mathcal{L}} + W R)U_I(\tau) + F(\tau), \quad (26)$$

The arising ODEs system can be numerically solved by common time stepping method or ode command of MATLAB such as `ode15s`.

V. NUMERICAL EXPERIMENTS

In this section, we demonstrate the performance of the RBF based approximation method for approximation of multi-asset option pricing. The implementation of the RBF approximation method and experiments have been divided into the two cases; two-asset and three-asset option. For both cases, we take the IMQ as a basis function, and wendland – for right function in RBF-PUM case. We discretize the operator in space by localization of the computational domain $\Omega_X = [-3, 3]^d$.

TABLE I
 PARAMETER VALUES USED FOR TWO-ASSET CASE

Diffusion		Jump		payoff	
parameter	value	parameter	value	parameter	value
σ_1	0.3	$\tilde{\sigma}_1$	0.4	r	0.05
σ_2	0.3	$\tilde{\sigma}_2$	0.4	T	1
ρ	1	$\tilde{\mu}_1$	0.1	E	1
		$\tilde{\mu}_2$	0.1	α_1	0.5
		$\tilde{\rho}$	0	α_2	0.5
		λ	0.1		

VI. TWO-ASSET OPTIONS

To ensure the accuracy of the RBF based methods for the multi-asset option pricing, we test the algorithm for a sample problem and compared with common Finite difference method. We use the FDM to generate reference solution with finer grids. The parameters for the jump diffusion model are given in Table I.

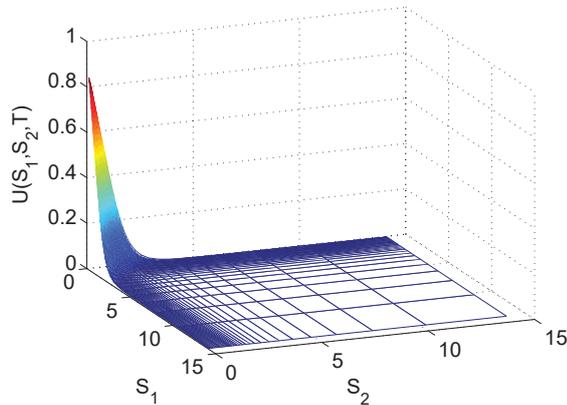


Fig. 1 Approximated solution of the basket option pricing based on Merton model with two underlying assets

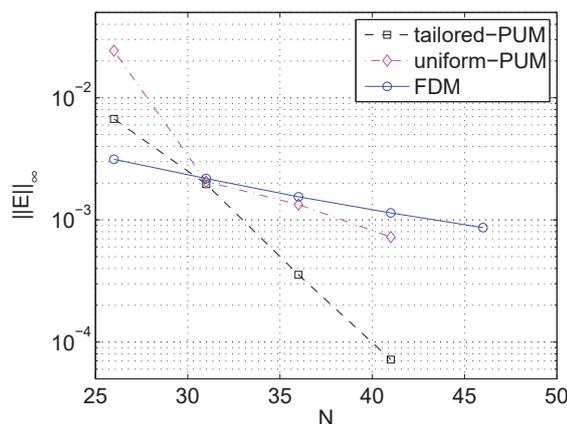


Fig. 2 Convergence behavior for RBF based method and FD method. Reference solution based on FD method with $N = 180$ grid in each axis is used for computation of solution

REFERENCES

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.* 81 (3) (1973) 637-654.

[2] R. C. Merton, Option pricing when underlying stock returns are discontinuous, *Journal of financial economics* 3 (1-2) (1976) 125-144.

[3] U. Pettersson, E. Larsson, G. Marcusson, J. Persson, Improved radial basis function methods for multi-dimensional option pricing, *J. Comput. Appl. Math.* 222 (1) (2008) 82-93.

[4] G. Fasshauer, A. Q. M. Khaliq, D. A. Voss, Using mesh free approximation for multi asset american options, in: C.S. Chen (Ed.), *Mesh free methods*, Journal of Chinese Institute of Engineers 27 (2004) 563-571, special issue.

[5] A. Safdari-Vaighani, A. Heryudono, E. Larsson, A radial basis function partition of unity collocation method for convection-diffusion equations arising in financial applications, *J. Sci. Comput.* 64 (2) (2015) 341-367.

[6] X. L. Zhang, Numerical analysis of American option pricing in a jump-diffusion model, *Math. Oper. Res.* 22 (3) (1997) 668-690.

[7] M. Briani, R. Natalini, G. Russo, Implicit-explicit numerical schemes for jump diffusion processes, *Calcolo* 44 (1) (2007) 33-57.

[8] R. Cont, E. Voltchkova, A finite difference scheme for option pricing in jump diffusion and exponential Levy models, *SIAM J. Numer. Anal.* 43 (4) (2005) 1596-1626 (electronic).

[9] Y. dHalluin, P. A. Forsyth, K. R. Vetzal, Robust numerical methods for contingent claims under jump diffusion processes, *IMA J. Numer. Anal.* 25 (1) (2005) 87-112.

[10] Y. dHalluin, P. A. Forsyth, G. Labahn, A penalty method for American options with jump diffusion processes, *Numer. Math.* 97 (2) (2004) 321-352.

[11] R. Brummelhuis, R. T. L. Chan, A radial basis function scheme for option pricing in exponential Levy models, *Appl. Math. Finance* 21 (3) (2014) 238-269.

[12] S. H. Martzoukos, Contingent claims on foreign assets following jump-diffusion processes, *Review of Derivatives Research* 6 (1) (2003) 27-45.

[13] S. S. Clift, P. A. Forsyth, Numerical solution of two asset jump diffusion models for option valuation, *Appl. Numer. Math.* 58 (6) (2008) 743-782.

[14] C. La Chioma, Integro-differential problems arising in pricing derivatives in jump-diffusion markets, Ph.D. thesis, PhD thesis, Rome University (2003).

[15] H. Windcliff, P. A. Forsyth, and K. R. Vetzal, Analysis of the stability of the linear boundary condition for the Black-Scholes equation, *Journal of Computational Finance*, 8 (2004) 65-92.

[16] E. Larsson, A. Heryudono, A partition of unity radial basis function collocation method for partial differential equations, manuscript in preparation (2016).