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A Markov Chain Approximation for ATS Modeling for the Variable Sampling Interval CCC Control Charts

Y. K. Chen, K. C. Chiou, and C. Y. Chen

Abstract**—**The cumulative conformance count (CCC) charts are widespread in process monitoring of high-yield manufacturing. Recently, it is found the use of variable sampling interval (VSI) scheme could further enhance the efficiency of the standard CCC charts. The average time to signal (ATS) a shift in defect rate has become traditional measure of efficiency of a chart with the VSI scheme. Determining the ATS is frequently a difficult and tedious task. A simple method based on a finite Markov Chain approach for modeling the ATS is developed. In addition, numerical results are given.

Keywords**—**Cumulative conformance count, variable sampling interval, Markov Chain, average time to signal, control chart.

I. Introduction

THE cumulative conformance count (CCC) chart is a decision model based on the number of items inspected until one nonconforming item is observed for monitoring the industrial process. Recently, it has shown particularly appropriate for the high-yield process in which the fraction of nonconforming is very slow at the level of part per million (ppm) or less [1]-[5].

Traditional CCC chart uses a single count value to detect changes in the fraction of nonconforming in the process. The merit of the CCC chart is simple, but it makes the chart relatively insensitive to small process changes [6]. To improve the performance of the chart, values of the previous runs or observations were incorporated into the decision rule using conditional probability [6]-[7]. Through numerical examples, they showed that the conditional chart had improved the performance relative to the traditional CCC chart.

Liu et al. [8] developed another decision procedure to improve the weakness. In their model, each inspected item is regarded as a sample, and the time between successively inspected items is treated as the sampling interval. Different from the standard CCC chart, the length of sampling interval varies depending on the region the geometric variate falls. The shorter sampling interval is used when there is some indication from the geometric variate showing the process may have changed; and the longer sampling interval is used when no such indication. The CCC chart with variable sampling intervals is called the CCC_{VSI} chart. The VSI scheme has been carried out in several studies of control charts to improve the sensitivity of detecting process disturbances without increasing the rate of inspected items and false alarm rate occurred [9]-[17].

In Liu et al's works [8], the detection property of the CCC_{VSI} chart was derived based on the approach of Wald's identity. This approach can be applied, but it is mathematically complicated for analysis. In the present paper, the CCC_{VSI} chart is approximately modeled as a Markov dependence process, and the detection property of the CCC_{VSI} chart is determined by the Markov chain approach, which standardizes the mathematics and simplifies the presentation.

Y. K. Chen is with the Logistics Engineering and Management Department, National Taichung Institute of Technology, 129 Sanmin Road, Sec. 3, Taichung, Taiwan (corresponding author to provide phone: 886-4-2376-2425; e-mail: ykchen@ntit.edu.tw).

K. C. Chiou is with the Finance Department, Chaoyang University of Technology, 168 Jifong E. Rd., Wufong Township Taichung County, 41349, Taiwan (e-mail: kcchiou.cyut@gmail.com).

C. Y. Chen is with the Electronic Engineering Department, National Yunlin University of Science & Technolgy, 123, Sec. 3, University Rd., Douliu, Yunlin, Taiwan (e-mail: chencyue@yuntech.edu.tw).

II. The CCC_{VSI} Chart

Consider a process in which the number of items inspected until one nonconforming item is observed is coming from a geometric distribution with parameter p_0. Let X stand for the above random number, then the probability mass function of X can be expressed by

$$P\{X = x\} = (1 - p_0)^{x-1} p_0 ,\ \ x = 1,2,\cdots \tag{1}$$

In a high-yield manufacturing environment, the values of p_0 is assumed very small.

When a CCC chart is applied to monitor p_0, the values of X are reckoned and plotted over time on the chart with the upper control limit (UCL) and lower control limit (LCL). Given an acceptable rate of false alarm α, the UCL and LCL can be determined by $p\{X \geq UCL\} = P\{X \leq LCL\} = \alpha/2$. Since the geometric distribution is discrete, the control limits are rounded to integers and calculated by

$$UCL = [\frac{\ln(\alpha/2)}{\ln(1 - p_0)} + 1] , \tag{2}$$

$$LCL = [\frac{\ln(1 - \alpha/2)}{\ln(1 - p_0)}] . \tag{3}$$

As p_0 is very small, the true false alarm rate getting from the rounded control limits is close to the acceptable one. The CCC chart signals a shift has occurred as soon as the $X \geq UCL$ or $X \leq LCL$.

For the CCC chart with fixed sampling interval (FSI), called the CCC_{FSI} chart, the time between two successive samples is fixed at h_f without any change through the process. In contrast, the CCC_{VSI} chart is divided into safety, warning and action regions, and the sampling interval for next sample is altered between two values, depending on the position of most recent X.

Fig. 1 depicted one example of the CCC_{VSI} charts. The first value X_1 in the figure falls within the safety region (WL, UCL), implying the process is running well and the nonconforming rate of the process p_0 may possibly remain unchanged. In this case, a long sampling interval h_1 is used for the next inspection of items to reduce the sampling cost. The second value X_2 is treated likewise. The third value X_3 falls within the warning region $(LCL, WL]$, implying the value of p_0 may possibly be increased due to some assignable cause while the next point X_4 has a large chance to fall below LCL. In this case, a long sampling interval h_2 is used for the next inspection of items to reduce the response time to the change. When the point such as X_{49} falls into the action region $[0, LCL] \cup [UCL, \infty)$, then an out-of-control signal will be alarmed to stop the process and to search the assignable cause. For simplicity we assumed the chart is started at time 0, and the first sampling interval uses the short sampling interval length h_2 because it gives additional protection against problems that arise during start-up in practice although no point is plotted at the process start (or restart after the assignable cause eliminated).

The warning limit (WL) is related to the probability allocation τ, given by the user. Since

$$\tau = \Pr\{LCL < X \le WL\} / \Pr\{LCL < X < UCL\}, \tag{4}$$

the warning limit can be determined by

$$WL = [\frac{\ln(1-\alpha/2-(1-\alpha)\tau)}{\ln(1-p_0)}]. \tag{5}$$

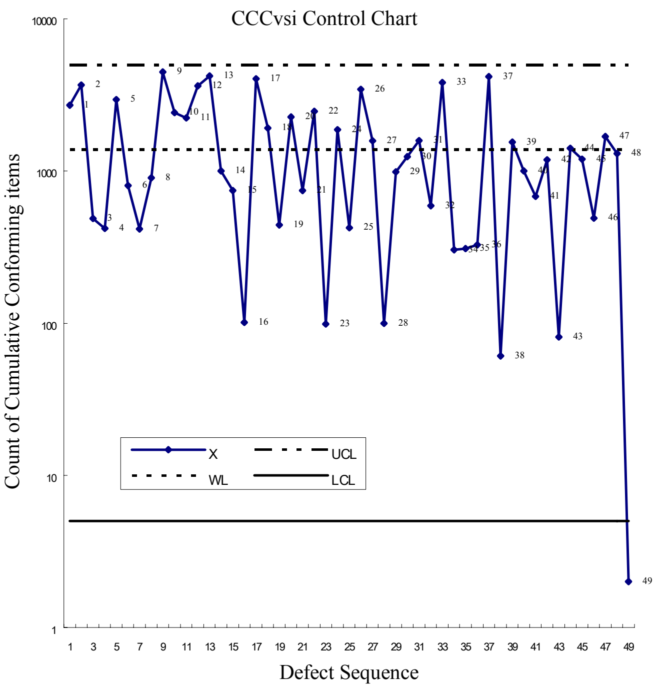

Fig. 1 Illustration of the CCC_{VSI} charts

III. Modeling the CCC_{VSI} Chart with Markov Chain

Operations of control charts can be considered as a graphical expression of statistical testing for a process problem, thus the false alarm rate (i.e., the probability of type I error) and the speed with which it detects the occurrence of the process problem (i.e., the power of testing) are often used to evaluate the performance of control charts. The ARL, defined as the average number of samples to signal (ANSS) an out-of-control condition, is the most widely used for comparing the statistical performance of different control charts when the intervals between samples are fixed and equal for the compared charts. The ARL should be long as the process is in-control, but short once a process mean shift occurs.

In the case of charts with the intervals between samples not fixed, the measure of average time to signal (ATS) an out-of-control condition would be an appropriate measure compared to ARL to evaluate the performance of control charts. The ATS is defined as the expected value of the time from the start of the process until the chart signals. The ATS is a multiple of the ARL when the sample size is fixed.

For any control chart with variable sampling intervals that is approximated by a Markov dependence process, the ATS performance can be determined from the Markov chain approach [18]. Derivation of ATS for the CCC_{VSI} chart using the Markov chain approach is illustrated as follows. Let

State 1 represents the state where $X \in (WL, UCL)$;

State 2 represents the state where $X \in (LCL, WL]$;

State 3 represents the state where $X \in [0, LCL] \cup [UCL, \infty)$. State 3 is an absorbing state.

At each point, one of the states 1 and 2 is reached according to the transition probability matrix for a nonconforming rate p, where $p = p_0$ means the in-control process nonconforming rate while $p = p_1$ means the out-of-control process nonconforming rate).

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

where

$p_{11} = p_{21} = \Pr\{WL < X < UCL\} = (1-p)^{WL} - (1-p)^{UCL-1}$;

$p_{12} = p_{22} = \Pr\{LCL < X \le WL\} = (1-p)^{LCL} - (1-p)^{WL}$;

$p_{13} = p_{23} = 1 - p_{11} - p_{12}$.

According to the elementary properties of the Markov chain, we know that $\mathbf{r}(\mathbf{I}-\mathbf{Q})^{-1}$ provides the mean number of transitions to each transient state before the true alarm signals, where $\mathbf{r} = (r_1, r_2)$ is the vector of starting probability such that $r_1 + r_2 = 1$ (to give additional protection against problems that arise during start-up, this vector can be set by $(0,1)$); $\mathbf{I}$ is the identity matrix of order 2; $\mathbf{Q}$ is the transition matrix $\mathbf{P}$ where elements associated with the absorbing state are deleted.

The product of the average number of visiting the transient state and the corresponding time until the next nonconformance appears determines the ATS. That is,

$$ATS = \mathbf{r}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{t} \tag{7}$$

where $\mathbf{t} = \left(\frac{h_1}{p}, \frac{h_2}{p}\right)$ is the vector of average time corresponding to the two transient states, required to observe the next nonconformance. Thus, ATS can be simply rewritten by

$$ATS = \left(\frac{p_{11}h_1 + (1-p_{11})h_2}{1-p_{11}-p_{12}}\right)\left(\frac{1}{p}\right). \tag{8}$$

In the FSI scheme the ATS can be easily obtained by letting $h_1 = h_2 = h_f$ and $WL = LCL$, which implies $p_{12} = p_{22} = 0$. In such case, the formula of (6) can be reduce as

$$ATS = \left(\frac{h}{1-p_{11}}\right)\left(\frac{1}{p}\right). \quad (9)$$

IV. Numerical Results

A. Selecting the Design Parameters

The CCC_{VSI} chart was compared with the CCC_{FSI} chart in terms of their statistical performances to evaluate its efficiency. The comparison between the VSI and FSI charts were conducted with the same nonconforming rate and acceptable false alarm rate to ensure the average number of inspected items is identical. Besides, the design parameters of the charts were selected in hopes of matching their in-control average time to signal (i.e., $ATS_0^V = ATS_0^F$) before comparison. The procedure of selecting the design parameters: h_1, h_2, WL, LCL, and UCL is suggested as follows.

*Step*0. Given h_f, α, p_0, and τ.

*Step*1. Determine LCL and UCL using Eqs. (2) and (3).

*Step*2. Determine WL using Eq. (5).

*Step*3. Generate the value for h_1 ($> h_f$) and thus the value for h_2 ($< h_f$) is determined by

$$h_2 = \frac{(1-p_{11}-p_{12})h_f}{(1-p_{11})(1-p_{11})} - \frac{p_{11}h_1}{1-p_{11}} \quad (10)$$

Note that it requires that h_2 chosen is not less than the time between two consecutive items.

B. Numerical Example

In this section, a numerical example is presented to illustrate the above procedure, and then the sensitivity analysis on the process parameters is made. The values for process parameters were partially selected from those in [8]. For example, we set α =0.0027, p_0 =0.0005, τ =0.5, and h_f =1 as well. Accordingly, LCL =2, UCL =3212, and WL =1385. if h_1 is set as 1.3 h_f, then h_2 will be 0.7 h_2. To compare with the FSI chart, the index of improvement by the VSI chart with respect to ATS_1 (the average time to signal when the process is out-of-control) is defined as

$$I = \frac{ATS_1^V}{ATS_1^F} \quad (11)$$

Table I shows the results of comparisons. As we have seen, the average time for the VSI chart to detect the process change is only about 85% of that of the FSI chart when the nonconforming rate of the process increases to twice more than the original level. In other words, there is a 15% improvement in the responding time to process change. The percentage improvement will increase when the difference between the lengths of h_1 and h_2 increases. Moreover, the index of improvement will decrease as the values of the ratio of p_1 to p_0 increased from 1.1 to 3.0. These results coincide with that in [8].

Table II continues the aforementioned example, and lists the index of improvement by the VSI chart against different combinations of α, p_0, τ, and degrees of out-of-control process nonconforming rate. From Table II, we can found the effects of α and p_0 on the index of improvement are almost negligible. As compared with α and p_0, the improvement in ATS_1 by the VSI chart is more remarkable. In addition, the larger the value for τ is, the larger the index of improvement is made.

V. Conclusion

Control charts based on cumulative count of conforming have been shown a useful tool in a high-quality process. The applicability of standard CCC charts is limited to the case that products from a process are sequentially inspected with fixed sampling interval, which may require more items inspected to detect the process change. In this paper we consider tracking cumulative count with variable sampling intervals to improve the efficiency of the standard CCC chart. The assumption of a geometric distribution to describe the cumulative counts allows the application of the Markov chain approach, which simplifies the results of Liu et al. [8]. According to the scheme of variable sampling intervals, we divided the CCC chart into three regions, the safety region, the warning region, and the action region. If the cumulative count of conforming falls in the warning region, we tighten the control by waiting less time before inspect next item. If the cumulative count of conforming falls in the safety region, then we relax the control by waiting longer to take next item. The efficiency of the CCC chart with variable sampling interval scheme is compared with the standard CCC chart. This comparison shows that the former outperforms the latter with respect to the average time to detect the increase of nonconforming rate. In addition, the improvement amplifies when the difference between the sampling lengths increases or when the increase of nonconforming rate becomes evident. From the results of sensitivity analysis, it is found that the influence of acceptable rate of false alarm as well as original nonconforming rate on the improvement is minor. Likewise, a warning limit with equal probability allocation seems to be most suitable for the division of the charts.

This study assumes the level of original nonconforming rate is given for the sake of simplicity. In practice, the true value of the process parameter, however, is unknown and required to estimate. Thus, it would be worthwhile conducting further research on the chart's property when the process parameter is estimated.

TABLE I
Index of Improvement with Different Lengths of VSI and Degrees of Out-of-Control Process Nonconforming Rate
(α =0.0027, p_0 =0.0005, τ =0.5) Magnetic Properties

p_1/p_0	(h_1, h_2)			
	(1.90, 0.10)	(1.70, 0.30)	(1.50, 0.50)	(1.30, 0.70)
1.0	1.00	1.00	1.00	1.00
1.1	0.94	0.95	0.97	0.98
1.2	0.89	0.91	0.94	0.96
1.3	0.83	0.87	0.91	0.94
1.4	0.78	0.83	0.88	0.93
1.5	0.74	0.80	0.86	0.91
1.6	0.70	0.76	0.83	0.90
1.7	0.66	0.73	0.81	0.89
1.8	0.62	0.70	0.79	0.87
1.9	0.59	0.68	0.77	0.86
2.0	0.55	0.65	0.75	0.85
3.0	0.33	0.48	0.63	0.78

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TABLE II
INDEX OF IMPROVEMENT WITH DIFFERENT VALUES OF α, p_0, τ, AND DEGREES OF OUT-OF-CONTROL PROCESS NONCONFORMING RATE
(FIXED h_1 =1.9)

				p_1/p_0											
α	p_0	τ	(h_1, h_2)	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	3.0
0.0010	0.0001	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.61)	1.00	0.96	0.92	0.88	0.85	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.10)	1.00	0.94	0.88	0.83	0.78	0.74	0.69	0.66	0.62	0.58	0.55	0.33
	0.0005	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.61)	1.00	0.96	0.92	0.88	0.85	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.10)	1.00	0.94	0.88	0.83	0.78	0.74	0.69	0.65	0.62	0.58	0.55	0.33
	0.009	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.61)	1.00	0.96	0.92	0.88	0.85	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.10)	1.00	0.94	0.88	0.83	0.78	0.74	0.69	0.66	0.62	0.58	0.55	0.33
0.0027	0.0001	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.62)	1.00	0.96	0.92	0.88	0.85	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.10)	1.00	0.94	0.89	0.83	0.78	0.74	0.70	0.66	0.62	0.59	0.55	0.33
	0.0005	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.62)	1.00	0.96	0.92	0.88	0.85	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.10)	1.00	0.94	0.89	0.83	0.78	0.74	0.70	0.65	0.62	0.59	0.55	0.33
	0.009	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.62)	1.00	0.96	0.92	0.88	0.85	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.10)	1.00	0.94	0.89	0.83	0.78	0.74	0.70	0.66	0.62	0.59	0.55	0.33
0.0050	0.0001	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.62)	1.00	0.96	0.92	0.89	0.86	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.11)	1.00	0.94	0.89	0.84	0.79	0.74	0.70	0.66	0.62	0.59	0.56	0.33
	0.0005	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.62)	1.00	0.96	0.92	0.89	0.86	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.11)	1.00	0.94	0.89	0.84	0.79	0.74	0.70	0.65	0.62	0.59	0.56	0.33
	0.009	0.9	(1.90,0.90)	1.00	0.98	0.96	0.95	0.94	0.93	0.93	0.92	0.92	0.91	0.91	0.90
		0.7	(1.90,0.62)	1.00	0.96	0.92	0.89	0.86	0.83	0.80	0.78	0.76	0.75	0.73	0.65
		0.5	(1.90,0.11)	1.00	0.94	0.89	0.84	0.79	0.74	0.70	0.66	0.62	0.59	0.56	0.33