Deduction of Fuzzy Autocatalytic Set to Omega Algebra and Transformation Semigroup

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Abstract**—**In this paper, the Fuzzy Autocatalytic Set (FACS) is composed into Omega Algebra by embedding the membership value of fuzzy edge connectivity using the property of transitive affinity. Then, the Omega Algebra of FACS is a transformation semigroup which is a special class of semigroup is shown.

Keywords**—**Fuzzy autocatalytic set, omega algebra, semigroup, transformation semigroup.

I. Introduction

THE concept of autocatalytic set (ACS) was first introduced in the context of catalytically interacting molecules [1, 2]. However, Jain and Krishna have formalized the autocatalytic set in terms of graph [3]. Tahir et. al. defined Fuzzy Autocatalytic Set (FACS) [4] as subgraph which each of those nodes has at least one incoming link with membership value $\mu(e_i) \in (0,1], \quad \forall\, e_i \in E$. The vertices of the graph correspond to the variables and a directed link from vertex *i* to vertex *j* indicates that variable *i* catalyzes the production of variable *j*. Fuzzy Graph Type 3, G_F^3 as a fuzzy graph where both the vertex and edge sets are crisp, but the edges have fuzzy heads and tails. Sabariah shown that the Fuzzy Graph Type-3 is a FACS [5] with the following definitions:

Definition 1 [4]:

Let $e_i \in E$. The fuzzy head of e_i denotes as $h(e_i)$ and the fuzzy tail $t(e_i)$ are functions of e_i such that $h: E \to [0,1]$ and $t: E \to [0,1]$ for $e_i \in E$. Fuzzy edge connectivity is a tuple $(t(e_i), h(e_i))$ and the set of all fuzzy edge connectivity is denoted as $C = \{(t(e_i), h(e_i)) : e_i \in E\}$.

The membership value of fuzzy edge connectivity is denoted as $\mu(e_i) = \min\{t(e_i), h(e_i)\}$.

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Definition 2 [5]:

Let $C_{F_{ij}}$ denoted the fuzzy edge connectivity between node *i* and node *j*, then:

$$C_{F_{ij}} = \begin{cases} 0 & ,\text{if } i = j \quad , e_i \notin E \\ \mu(e_i) & ,\text{if } i \neq j \end{cases}$$

II. Omega Algebra and Graph

An operation is an action or procedure which produces a new value from one or more input value. *N*-ary algebraic operation on the set M is a function of the form $\omega_n : M_1 \times M_2 \times \ldots \times M_n \to M$. In others word, the *n*-ary or arity of a function or operation is the number of argument or operands of the function. In the function $\omega_n : M^n \to M$, for some set *M* is an operation and *n* is its arity. Therefore, omega algebra (Ω-algebra) is a set with the certain system of operations Ω i.e. Ω-algebra$=\{\omega_k \mid k = \{0,1,\ldots n\}\}$ which is defined on one basic set and is called one-sorted algebraic system [6]. For example

$$\omega_2 : M \times M \to M$$

$$\omega_3 : M \times M \times M \to M$$

$$\vdots$$

$$\omega_n : M \times M \times \cdots \times M \to M$$

When a system is defined as a complete directed graph, it means for *any* two elements in the system, there exist two *connections* between them, i.e. v_i connects to v_j *and* v_j connects to v_i and $M_i = M_j = M$. The term 'connection' here depends on the relevancy of system one going to define. Thus, if the Ω-algebra is applied in the completed directed graph, it is interpreted as *the mapping of the Cartesian product of any set of vertices to one of its vertex.* Ω-algebra was physically interpreted as "for any *n* element in an arbitrary system, the connection between these *n* elements will produce one of these elements as a final product". Figure 1 gives the illustration in explaining the omega algebra of five elements. For example, $M = \{v_1, v_2, v_3, v_4, v_5\}$ for any $v_i \in M$, the Ω-algebra$=\{\omega_k \mid k = \{2,3,4,5\}\}$ for

$\omega_2 : v_i \times v_j \to v_k$, $\omega_3 : v_i \times v_j \times v_m \to v_k$, $\omega_4 : v_i \times v_j \times v_m \times v_n \to v_k$ and $\omega_5 : v_i \times v_j \times v_m \times v_n \times v_x \to v_k$. There is no repetition of vertex is assumed in this system.

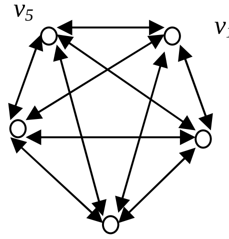

Fig. 1 Ω - algebra for M

III. Omega Algebra of FACS

The hierarchical relationship between cycles, irreducible subgraph and ACSs is shown by Jain and Krishna [3, 7] as below:

Proposition 1[3, 7]:
PI: All cycles are irreducible subgraphs and all irreducible subgraphs are ACSs.
PII: Not all ACSs are irreducible subgraphs and not all irreducible subgraphs are cycles.

FACS is defined by introducing fuzzy concept into the autocatalytic set, hence, Proposition I and II also hold for FACS:

Proposition 2:
PFI: All cycles are irreducible subgraphs and all irreducible subgraphs are FACSs
PFII: Not all FACSs are irreducible subgraphs and not all irreducible subgraphs are cycles.

As in PFII, that not all FACS are irreducible subgraph. Further, we let our FACS to be an irreducible FACS (*i*FACS). This leads to the following terms.

(i) $V_{i\text{FACS}} = \{v_i \mid 1 \le i \le n\}$ is a set of vertices of *i*FACS which every vertex have at least one incoming edges with membership value $\mu(e_i) \in (0,1], \forall e_i \in E$ from one vertex belonging to the same *i*FACS.

(ii) $\omega_{k(v_i, v_j)}$ represents "v_i *and* v_j *have k-ary relation which* v_i *catalyzed the production of* v_j *through k-ary Cartesian product of* * *operation*" or "*there exist some* v_i *and* v_j *such that* v_i *and* v_j *have k-ary catalytic relation in* ω*-operation*".

Theorem 1:
The set of $\omega -$operations of *i*FACS, form the Ω - algebra of *i*FACS i.e. $\Omega_{i\text{FACS}} = \{\omega_k \mid k = 1, 2, \ldots, n\}$.

Proof:
*i*FACS is a FACS by PI and PFI. As in FACS, the simplest FACS is 1-cycle [3], which is a loop and this 1-cycle FACS is also an *i*FACS. Suppose that $V_{i\text{FACS}}$ is the set of M as in Section II, thus $\omega -$operations that exist in the set of $V_{i\text{FACS}}$ are:

- unary operation: $\omega_1 : V_{i\text{FACS}} \to V_{i\text{FACS}}$ such that $\exists v_i \in V_{i\text{FACS}} \ni \omega_{1(v_i, v_i)} = v_i \in V_{i\text{FACS}}$,
- binary operation: $\omega_2 : V_{i\text{FACS}} \times V_{i\text{FACS}} \to V_{i\text{FACS}}$ such that for $v_i, v_j \in V_{i\text{FACS}}$, $\omega_{2(v_i, v_j)} = v_j \in V_{i\text{FACS}}$,
- ternary operation: $\omega_3 : V_{i\text{FACS}} \times V_{i\text{FACS}} \times V_{i\text{FACS}} \to V_{i\text{FACS}}$ or $\omega_3 : V^3_{i\text{FACS}} \to V_{i\text{FACS}}$ such that for $v_i, v_j, v_k \in V_{i\text{FACS}}$, $\omega_{3(v_i, v_k)} = v_k \in V_{i\text{FACS}}$ through v_j,

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- n-ary operation: $\omega_n : V^n_{i\text{FACS}} \to V_{i\text{FACS}}$ such that $\exists v_i, v_p \in V_{i\text{FACS}}$, $\omega_{n(v_i, v_p)} = v_p \in V_{i\text{FACS}}$.

This set of $\omega -$operations form the omega algebra of *i*FACS, i.e. $\Omega_{i\text{FACS}} = \{\omega_k \mid k = 1, 2, \ldots, n\}$. □

In this Ω - algebra representation of *i*FACS, $\Omega_{i\text{FACS}} = \{\omega_k \mid k = 1, 2, \ldots, n\}$ is a set of Ω - algebra operations in which

1. is a *set of path* from a vertex to itself in the length of k or to another vertex through the path with length of $k-1$.
2. is a *set of omega operations* that physically means catalytic relation among the element of *i*FACS.

Next, the $(\Omega_{i\text{FACS}}, \Theta)$ is a semigroup is shown with the following definition of the binary operation:

Definition 3:
An operation Θ is defined for $\Omega_{i\text{FACS}}$ in which when $v_i, v_j, v_p, v_q \in V_{i\text{FACS}}$ for $x, y \in \{1, 2, 3, .., n\}$ then

$\omega_x \Theta \omega_y = \omega_{x(v_i,v_j)} \Theta \omega_{y(v_p,v_q)} = v_j \Theta v_q$
$= \omega_{s(v_j,v_q)} \in \omega_s \in \Omega_{i\text{FACS}}$, for some $s \in \{1,2,3,...,n\}$
where $\omega_{s(v_j,v_q)} = v_q$.

Theorem 2:
$(\Omega_{\text{iFACS}},\Theta)$ is a semigroup.

Proof:
Let $\Omega_{i\text{FACS}}=\{\omega_k \mid k=1,2,\ldots,n\}$.
Consider $\omega_{x(v_i,v_j)} = v_j$, $\omega_{y(v_g,v_h)} = v_h$ and $\omega_{z(v_p,v_q)} = v_q$.

$$\begin{aligned}\omega_x\Theta\omega_y &= \omega_{x(v_i,v_j)}\Theta\omega_{y(v_g,v_h)}\\ &= v_j\Theta v_h\\ &= \omega_{r(v_j,v_h)} \in \Omega_{i\text{FACS}} \text{ for some } r.\end{aligned}$$

Next,

$$\begin{aligned}(\omega_x \Theta \omega_y)\Theta \omega_z &= (\omega_{x(v_i,v_j)} \Theta \omega_{y(v_g,v_h)})\Theta \omega_{z(v_p,v_q)}\\ &=(v_j,v_h)\Theta \omega_{z(v_p,v_q)}\\ &=\omega_{s(v_j,v_h)}\Theta\omega_{z(v_p,v_q)} \text{ for some } s\\ &=v_h \Theta v_q\\ &=\omega_{t(v_h,v_q)} \text{ for some } t\\ &=v_q\end{aligned}$$

and

$$\begin{aligned}\omega_x\Theta(\omega_y \Theta \omega_z) &= \omega_{x(v_i,v_j)} \Theta (\omega_{y(v_g,v_h)} \Theta \omega_{z(v_p,v_q)})\\ &=\omega_{x(v_i,v_j)} \Theta (v_h\Theta v_q)\\ &=\omega_{x(v_i,v_j)}\Theta\omega_{r(v_h,v_q)} \text{ for some } r\\ &=v_j \Theta v_q\\ &=\omega_{w(v_j,v_q)} \text{ for some } w\\ &=v_q\end{aligned}$$

Hence, $(\omega_x \Theta \omega_y)\Theta \omega_z = \omega_x \Theta (\omega_y \Theta \omega_z)$ □

The closure of Θ means a catalytic reaction will produce an element which is one of the chemical elements or variables in the clinical waste incineration process [4]. Furthermore, the associativity implies that it is free in regards to the order of the catalytic reactions.

Lemma 1:
$(\Omega_{\text{iFACS}},\Theta)$ is reflexive.
Proof:

$$\begin{aligned}\omega_x\Theta\omega_x &= \omega_{x(v_i,v_j)}\Theta\omega_{x(v_i,v_j)}\\ &= v_j \Theta v_j\\ &= \omega_{s(v_j,v_j)} \text{ for some } s\\ &= v_j\end{aligned}$$ □

Lemma 2:
$(\Omega_{\text{iFACS}},\Theta)$ is transitive.
Proof:
Let $\omega_x, \omega_y, \omega_z \in \Omega_{i\text{FACS}}$, therefore

$$\begin{aligned}\omega_x \Theta \omega_y &= \omega_{x(v_i,v_j)} \Theta \omega_{y(v_p,v_q)}\\ &= v_j \Theta v_q\\ &= \omega_{m(v_j,v_q)} \text{ for some } m\\ &= v_q\end{aligned}$$

and

$$\begin{aligned}\omega_y \Theta \omega_z &= \omega_{y(v_p,v_q)} \Theta \omega_{z(v_g,v_h)}\\ &= v_q \Theta v_h\\ &= \omega_{n(v_q,v_h)} \text{ for some } n\\ &= v_h\end{aligned}$$

Thus,

$$\begin{aligned}\omega_x \Theta \omega_z &= \omega_{x(v_i,v_j)} \Theta \omega_{z(v_g,v_h)}\\ &= v_j \Theta v_h\\ &= \omega_{r(v_j,v_h)} \text{ for some } r\\ &= v_h\end{aligned}$$ □

Unfortunately, $(\Omega_{\text{iFACS}},\Theta)$ does not observe the symmetry property. Let $\omega_x, \omega_y \in \Omega_{i\text{FACS}}$, then

$$\begin{aligned}\omega_x\Theta \omega_y &= \omega_{x(v_i,v_j)}\Theta\omega_{y(v_p,v_q)}\\ &= v_j \Theta v_q\\ &= \omega_{s(v_j,v_q)} \text{ for some } s\\ &= v_q\end{aligned}$$

and

$$\begin{aligned}\omega_y\Theta \omega_x &= \omega_{y(v_p,v_q)}\Theta \omega_{x(v_i,v_j)}\\ &= v_q\Theta v_j\\ &= \omega_{t(v_q,v_j)} \text{ for some } t\\ &= v_j\end{aligned}$$

Hence, Θ is not a symmetry since $\omega_x\Theta\omega_y \neq \omega_y\Theta\omega_x$.

Consequently, $(\Omega_{\text{iFACS}},\Theta)$ is not an equivalence relation.

Selection of desirable membership value of fuzzy connectivity of $\omega-$operations is subject to its application. If $\omega_{k(v_i,v_j)} \in \Omega_{i\text{FACS}}$, the membership value of fuzzy edge connectivity for $\omega_{k(v_i,v_j)}$ is denoted as $\mu(\omega_{k(v_i,v_j)})$ such that $\mu(\omega_{k(v_i,v_j)}) \in (0,1]$.

Suppose v_1 catalysts the production of v_2 and v_2 catalysts the production of element v_3, it is likely that element v_1 also catalysts the production of element i.e. there is a nonzero transitive affinity between v_1 and v_3 [8]. Since the FASC is defined as an irreducible FACS previously, thus, every vertex is being connected to another vertex in the same FACS.

The associativity of $\Omega_{i\text{FACS}}$ is consistent with transitive affinity. In other words, fuzzy edge connectivity is the weakest link for a particular path. In general, for any different paths that connecting any two vertices, fuzzy edge connectivity of a path which is maximal among those paths is chosen to be the membership value for fuzzy edge connectivity of that pair of vertices.

Definition 4:

Let $\omega_{x(v_i,v_j)} \in \Omega_{i\text{FACS}}$ i.e. path of $(v_i, v_{k_1}, \ldots, v_{k_{x-2}}, v_j)$. The membership value for fuzzy edge connectivity of $\omega_{x(v_i,v_j)}$ is defined as $\mu(\omega_{x(v_i,v_j)}) = \min(\varpi_{2(ij)})$ where $\varpi_{2(ij)} \equiv \{\mu(\omega_{2(v_i,v_{k_1})}), \mu(\omega_{2(v_{k_1},v_{k_2})}), \ldots, \mu(\omega_{2(v_{k_{x-3}},v_{k_{x-2}})}), \mu(\omega_{2(v_{k_{x-2}},v_j)})\}$.

Definition 5:

The maximal membership value of fuzzy edge connectivity between v_i and v_j is defined as $\mu(\omega_{ij}) = \max_{\omega_{x(v_i,v_j)}} \omega_{x(v_i,v_j)}$, where $\omega_{x(v_i,v_j)}$ is any possible path between v_i and v_j.

Therefore, for a given *i*FACS, the membership value of fuzzy edge connectivity between any pair of vertices is uniquely defined. Hence, for $k = 1, 2, \ldots n$, *i*FACS with order of j with their unique membership value of fuzzy edge connectivity obtained in the following table:

TABLE I
MEMBERSHIP VALUE OF FUZZY EDGE CONNECTIVITY BETWEEN ANY PAIR OF VERTICES

	v_1	v_2	$\cdots$	v_j
v_1	$(\omega_{k(v_1,v_1)}, \mu(\omega_{11}))$	$(\omega_{k(v_1,v_2)}, \mu(\omega_{12}))$	$\cdots$	$(\omega_{k(v_1,v_j)}, \mu(\omega_{1j}))$
v_2	$(\omega_{k(v_2,v_1)}, \mu(\omega_{21}))$	$(\omega_{k(v_2,v_2)}, \mu(\omega_{22}))$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
v_j	$(\omega_{k(v_j,v_1)}, \mu(\omega_{j1}))$	$(\omega_{k(v_j,v_2)}, \mu(\omega_{j2}))$	$\cdots$	$(\omega_{k(v_j,v_j)}, \mu(\omega_{jj}))$

IV. TRANSFORMATION SEMIGROUP OF Ω_{FACS}

It is obvious that $\Omega_{i\text{FACS}} \subseteq \Omega_{\text{FACS}}$ since all irreducible subgraphs are FACSs. With the proven definition of semigroup of $\Omega_{i\text{FACS}} = \{\omega_k \mid k = 1, 2, \ldots, n\}$, the transformation semigroup of Ω_{FACS} will be examined.

Definition 6 [9]:

A transformation semigroup, $X = (Q, S)$ which consist of a finite set *Q* and a subsemigroup *S* of *PF(Q)*. The elements of *Q* are called states, and *Q* itself is called the underlying set of *X*. The elements of *S* are called transformations of *X*, while *S* itself is called the action semigroup of *X* (see Fig. 2).

Notice that *PF*(*Q*) is a partial function over *Q* i.e. $Q' \to Q$ where $Q' \subseteq Q$ and (S, Θ) is a semigroup.

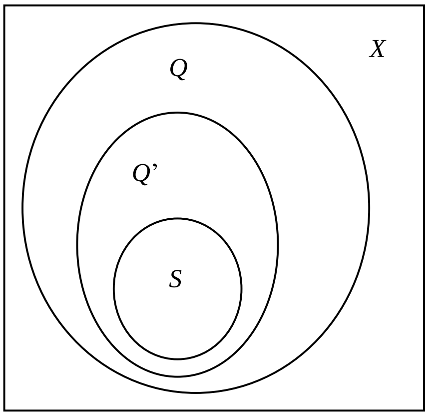

Fig. 2 Transformation semigroup respective sets

Theorem 3:

A transformation semigroup of Ω_{FACS}, $ts\Omega_{\text{FACS}}$ is a tuple (Ω_{iFACS}, S) where $\Omega_{i\text{FACS}}$ is called state or the underlying set of Ω_{FACS}. S is called transformation of Ω_{FACS}, while S itself is called the action semigroup of Ω_{FACS}.

Proof:

Recall, let Θ defined on $\Omega_{i\text{FACS}}$, such that for any element $\omega_x, \omega_y \in \Omega_{i\text{FACS}}$, then $\omega_x \Theta \omega_y \in \Omega_{i\text{FACS}}$. The (Ω_{iFACS}, S) is shown to be a semigroup.

Next, $\Omega_{i\text{FACS}}$ is a set of path of all *i*FACS and $\Omega_{i\text{FACS}} \subseteq \Omega_{\text{FACS}}$ is revealed. Θ is a function $\Theta : \Omega_{i\text{FACS}}' \to \Omega_{i\text{FACS}}$. But $\Omega_{i\text{FACS}}' = \Omega_{i\text{FACS}}$ is acknowledged since $(\Omega_{i\text{FACS}}, \Theta)$ is closed. Consider α such that $\alpha : \Omega_{i\text{FACS}} \times S \to \Omega_{i\text{FACS}}$ which is compatible with the semigroup operation $\Theta : \Omega_{i\text{FACS}} \times \Omega_{i\text{FACS}} \to \Omega_{i\text{FACS}}$ as follow: For all $s, t \in S$ and ω_x, $s\alpha(t\alpha\omega_x) = (s\Theta t)\alpha\omega_x$. With the above implementations, we have the $ts\Omega_{\text{FACS}}$. □

V. Conclusion

Fuzzy Autocatalytic Set can be composed into Omega Algebra is established. Then the structure of FACS is further extended into transformation semigroup of FACS.

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