

# The Study of Increasing Environmental Temperature on the Dynamical Behaviour of a Prey-Predator System: A Model

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**Abstract**—It is well recognized that the green house gases such as Chlorofluoro Carbon (*CFC*),  $CH_4$ ,  $CO_2$  etc. are responsible directly or indirectly for the increase in the average global temperature of the Earth. The presence of *CFC* is responsible for the depletion of ozone concentration in the atmosphere due to which the heat accompanied with the sun rays are less absorbed causing increase in the atmospheric temperature of the Earth. The gases like  $CH_4$  and  $CO_2$  are also responsible for the increase in the atmospheric temperature. The increase in the temperature level directly or indirectly affects the dynamics of interacting species systems. Therefore, in this paper a mathematical model is proposed and analysed using stability theory to assess the effects of increasing temperature due to greenhouse gases on the survival or extinction of populations in a prey-predator system. A threshold value in terms of a stress parameter is obtained which determines the extinction or existence of populations in the underlying system.

**Keywords**—Equilibria, Green house gases, Model, Populations, Stability.

## I. INTRODUCTION

THE atmospheric concentration of greenhouse gases such as carbondioxide, chlorofluoro carbon (*CFC*), methane and nitrous oxide is increasing due to rapid industrialization, extensive use of automobiles and burning of fossil fuels, construction of power plants and other anthropogenic activities. The excess of *CFC* depletes the ozone layer in the atmosphere and therefore the heat accompanied with the sun rays are less absorbed due to which the atmospheric temperature of the earth is increasing. The greenhouse gases like  $CH_4$  and  $CO_2$  are directly responsible for the increase in the atmospheric temperature. In recent years researchers, [2], [3] have predicted that the greenhouse gases will dramatically change global weather pattern in the next century and temperature of the Earth will rise in the years to come. Now, it is evident that the average global temperature of the Earth is rising due to increasing greenhouse gases. [4] provide an intriguing example of how one of the fundamental food-web properties that is connectance-changes with increasing temperature. In this paper a theoretical model based on assumptions from metabolic theory and foraging biology has been studied and it has been shown that the increasing temperature would have large effects on connectance with further consequences on the stability of interacting species systems. [5], [6] focus on how climatic

warming affects the metabolic rate of organisms, that is, the power required to sustain them and how these changes in metabolism scale up to ecosystem processes. It has been shown in the papers of [7] and [8] that the increasing temperature changes the processes at different levels of biological organization. [9] has shown that species higher in the food web like top predators tend to be more sensitive to temperature change. [10] has found that top predators moving towards cooler climates may trigger trophic cascades and coextinctions may also occur. Thus, it is noticed that the increase in temperature level may directly or indirectly effect the dynamics of interacting species systems. Therefore, it is essential to assess mathematically the effects of increasing  $CO_2$ , *CFC* and  $CH_4$  on populations in order to take necessary measures to avoid any adverse impact on an ecosystem. For understanding the consequences of the greenhouse effects on ecosystem an investigation of the interspecific interactions within biotic communities is required. A very few models to study temperature dependent interacting species systems exist [11], [12], [13], [14], [15], [16], [17], [18], [19], [20].

In view of the above, therefore in this paper, a mathematical model has been proposed and analyzed to study the effects of increasing temperature due to greenhouse gases on the survival or extinction of the populations in a prey-predator system. In the model it is assumed that the temperature increases directly due to greenhouse gases  $CH_4$  and  $CO_2$  and indirectly due to the greenhouse gas *CFC* (Chlorofluoro carbon) because of the depletion of the ozone concentration in the atmosphere by *CFC*. In the model it is further assumed that the rise in temperature negatively effects the intrinsic growth rate of the prey and adversely effects the prey-predator interaction phenomenon.

## II. MATHEMATICAL MODEL

Let  $N_1$  denote the density of a prey population which is growing logistically and  $N_2$  denote the density of a predator population.  $C$  denotes the concentration of *CFC* (Chlorofluoro carbon).  $C_1$  denotes the combined concentration of  $CH_4$  and  $CO_2$ .  $Z$  denotes ozone concentration. We consider here that  $T$  is elevated temperature or average increased temperature of the surrounding environment where the species live.

For a predator let its searching time per unit prey is  $d_1$ . Hence, searching time of the predator population for prey density  $N_1$  is  $d_1 N_1$ . If the handling time spent per unit prey by a predator is  $d_2$  then, the handling time for prey density  $N_1$  is  $d_2 N_1$ .

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It is assumed in the model construction that the searching time is adversely affected by the increasing temperature of the environment and therefore  $d_1$  is taken to be  $d_1/(1+B(T-T_0))$ . Similarly, it is also assumed in the model formulation that the handling time is also adversely affected by the increasing temperature of the environment and therefore,  $d_2$  is considered to be  $d_2/(1+B(T-T_0))$ . Thus, the total searching and handling time by a predator for prey density  $N_1$  is given by

$$\frac{d_1 N_1}{1+B(T-T_0)} + \frac{d_2 N_1}{1+B(T-T_0)} = \frac{(d_1+d_2)N_1}{1+B(T-T_0)} = \frac{a_1 N_1}{1+B(T-T_0)} \quad (1)$$

where,  $a_1 = d_1 + d_2$ .

From the expression (1), we note that when the environment is at the normal temperature;  $T_0$ , that is, at  $T = T_0$ , the predator behaves naturally and there is no change in their searching and handling time. We also notice from (1) that the predation rate will only be affected when temperature  $T$  exceeds  $T_0$ . With the above notations and assumptions, the mathematical model of the system under consideration is given by the following system of nonlinear differential equations:

$$\frac{dN_1}{dt} = r_1(T)N_1 - \frac{a_1 N_1 N_2}{1+B(T-T_0)} - \frac{r_{10} N_1^2}{K_{10}}, \quad (2)$$

$$\frac{dN_2}{dt} = -r_{20} N_2 + \frac{a_2 N_1 N_2}{1+B(T-T_0)}, \quad (3)$$

$$\frac{dC}{dt} = P - \frac{C}{\tau} - \beta CZ, \quad (4)$$

$$\frac{dC_1}{dt} = I_0 - \frac{C_1}{\tau_1}, \quad (5)$$

$$\frac{dZ}{dt} = Q_0 - \alpha_2 Z - \beta ZC, \quad (6)$$

$$\frac{dT}{dt} = h(C_1 - C_{10}) + \frac{K_1}{K_2 + Z} - \alpha_1(T - T_0), \quad (7)$$

with the initial conditions as:

$N_1(0) > 0, N_2(0) > 0, C(0) \geq 0, C_1(0) \geq 0, Z(0) > 0, T(0) > 0$

In the present analysis we assume the following form of  $r_1(T)$ :

$$r_1(T) = r_{10} - r_{11}(T - T_0), \quad r_1(T) > 0 \forall T, \quad r_1(T_0) = r_{10} \quad (8)$$

and  $a_2 = \gamma a_1$  where  $\gamma$  is conversion coefficient. It may be noted here that at the normal temperature  $T_0$ , the growth rate of prey population is  $r_{10}$  which is its intrinsic growth rate. The system parameters are defined as follows:

$r_{20}$  is natural death rate of predator population.  $B$  is a constant which measures the stress of temperature on both searching and handling time.  $I_0$  is input rate of  $C_1$ .  $C_{10}$  is the natural atmospheric concentration of  $C_1$ .  $P$  is input rate of  $C$ .  $\tau$  is average atmospheric residence time of  $CFC$ .  $\tau_1$  is average atmospheric residence time of  $C_1$  ( $CH_4$  and  $CO_2$  taken together).  $\beta$  is the depletion rate of ozone due to  $CFC$ .  $h$  is proportionality constant.  $Q_0$  is the natural formation rate of ozone concentration in the atmosphere.  $T_0$  is average normal temperature of earth surface of the area

occupied by the populations under study.  $r_1(T)$  is growth rate of population  $N_1$ .  $K_{10}$  is carrying capacity of prey population  $N_1$ .  $\alpha_1$  is coefficient of surface heat transfer and  $\alpha_2$  is natural depletion rate of ozone concentration. Here, all the parameters  $K_1, K_2, r_{10}, K_{10}, r_{11}, B, a_1, a_2, I_0, Q_0, \beta, \alpha_1, \alpha_2, h$  and  $r_{20}$  are taken to be positive constants.

### III. BOUNDEDNESS AND DYNAMICAL BEHAVIOUR

In this section we will establish that the solutions of the model given by the set of equations (2) to (7) with equation (8) are bounded in  $R^6_+$ . The boundedness of solutions is given by the following lemma.

**Lemma 3.1:** *All the solutions of the model will lie in the region  $V_1 = \{(N_1, N_2, C, C_1, Z, T) \in R^6_+ : 0 < N_1 \leq K_{10}, 0 < N_2 \leq r_{10}K_{10}/\eta, 0 < C \leq C_M, 0 < C_1 \leq C_{1M}, 0 < Z_m \leq Z \leq Z_M, 0 < T_m \leq T \leq T_M\}$ , as  $t \rightarrow \infty$ , for all positive initial values  $(N_1(0), N_2(0), C(0), C_1(0), Z(0), T(0)) \in R^6_+$ , where,  $C_M = P\tau, C_{1M} = (I_0 + Q_0)/\eta, \eta = \min(1/\tau_1, \alpha_2)$ ,*

*$T_M = \frac{hI_0\tau_1}{\alpha_1} + \frac{K_1}{\alpha_1(K_2+Z_m)} + T_0, T_m = \frac{\frac{K_1}{K_2+Z_m} - hC_{10}}{\alpha_1} > 0$  if  $\frac{K_1}{K_2+Z_M} > hC_{10}$*

*$Z_M = Q_0/\alpha_2$  and  $Z_m = Q_0/(\alpha_2 + \beta P\tau)$ .*

(for proof see Appendix A)

We now find all the feasible equilibria of the system (2) - (7). The system of equations (2) - (7) has three feasible equilibria  $E_i (i = 1, 2, 3)$  as given below:

1.  $E_1(N_1^*, N_2^*, C^*, C_1^*, Z^*, T^*)$ ,

where,  $N_1^* = 0, N_2^* = 0$ ,

$$C^* = \frac{P\tau}{1 + \beta\tau Z^*}, \quad (9)$$

$$C_1^* = I_0\tau_1, \quad (10)$$

$$Z^* = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_1}, \quad (11)$$

$a_1 = \alpha_2\beta\tau$ ,

$a_2 = \alpha_2 + \beta\tau(P - Q_0)$ ,

$a_3 = -Q_0$ ,

$$T^* = \frac{1}{\alpha_1} \left[ h(C_1^* - C_{10}) + \frac{K_1}{K_2 + Z^*} + \alpha_1 T_0 \right]. \quad (12)$$

The equilibrium  $E_1$  exists if  $I_0\tau_1 > C_{10}$ .

2.  $E_2(N_1^*, N_2^*, C^*, C_1^*, Z^*, T^*)$ ,

where,

$$N_1^* = \frac{r_1(T^*)K_{10}}{r_{10}},$$

$N_2^* = 0$  and

$C^*, C_1^*, Z^*, T^*$  are given by (9) - (12) respectively.

The equilibrium  $E_2$  exists if  $I_0\tau_1 > C_{10}$  and  $r_1(T^*) > 0$ .

3.  $E_3(N_1^*, N_2^*, C^*, C_1^*, Z^*, T^*)$ ,

where,

$$N_1^* = \frac{r_{20}}{\gamma a_1} (1 + B(T^* - T_0)),$$

$$N_2^* = \frac{1 + B(T^* - T_0)}{K_{10}\gamma a_1^2} [r_1(T^*)K_{10}\gamma a_1 - r_{10}r_{20}(1 + B(T^* - T_0))]$$

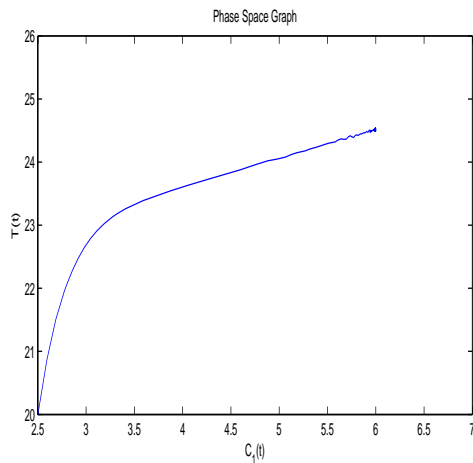


Fig. 1. Phase space diagram for concentration of greenhouse gases  $C_1(t)$  and temperature  $T(t)$ .

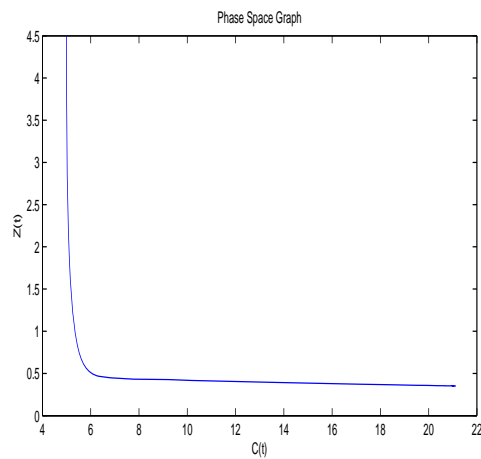


Fig. 3. Phase space diagram for concentration of chlorofluoro carbon  $C(t)$  and ozone concentration  $Z(t)$ .

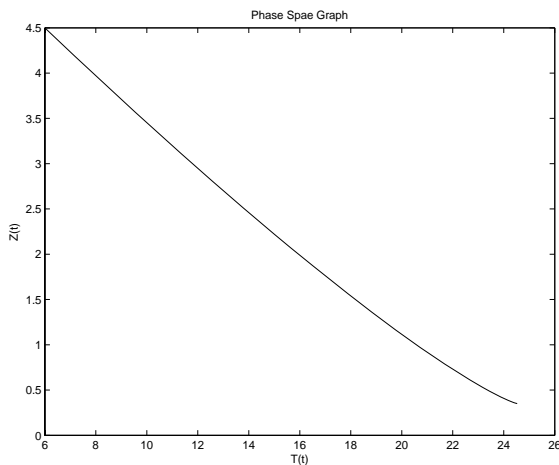


Fig. 2. Phase space diagram for temperature  $T(t)$  and ozone concentration  $Z(t)$ .

and  $C^*, C_1^*, Z^*, T^*$  are given by (9) - (12) respectively. The equilibrium  $E_3$  exists if  $I_0\tau_1 > C_{10}$ ,  $T^* > T_0$  and

$$\frac{r_{10}}{K_{10}} > \frac{\gamma a_1 r_{11} (T^* - T_0)}{\gamma a_1 K_{10} - r_{20} (1 + B(T^* - T_0))} \quad (13)$$

**Remark:** From the equilibrium value it is noted that the environmental temperature increases on account of increasing  $CH_4$  and  $CO_2$  concentrations (see Fig.1) and decreasing ozone concentration in the atmosphere (see Fig.2). Further, it may be noted that the equilibrium ozone concentration decreases due to the increase in the equilibrium concentration of chlorofluoro-carbon (see Fig.3).

Now we discuss the dynamical behaviour of the model by conducting the local and global stability analysis of the equilibria of the model.

#### A. Local Stability

The characteristic equation associated with the variational matrix about equilibrium  $E_1$  is given by

$$(J_1 - \lambda)(J_2 - \lambda)(J_5 - \lambda)(J_8 - \lambda)\{(J_7 - \lambda)(J_3 - \lambda) - J_4 J_6\} = 0, \quad (14)$$

where,  $J_1 = r_1(T^*)$ ,  $J_2 = -r_{20}$ ,  $J_3 = -1/\tau - \beta Z^*$ ,  $J_4 = -\beta C^*$ ,  $J_5 = -1/\tau_1$ ,  $J_6 = -\beta Z^*$ ,  $J_7 = -(\alpha_2 + \beta C^*)$ ,  $J_8 = -\alpha_1$

From the nature of the roots of the characteristic equation (14) we observe that the equilibrium point  $E_1$  is locally unstable provided  $r_1(T^*) > 0$ .

**Remark:** If  $r_1(T^*) < 0$ , then  $E_1$  is locally asymptotically stable and obviously both prey and predator populations would die out eventually.

The characteristic equation related to the equilibrium point  $E_2$  is obtained as

$$(G_2 - \lambda)(G_5 - \lambda)(G_1 - \lambda)(G_8 - \lambda)\{(G_3 - \lambda)(G_7 - \lambda) - G_4 G_6\} = 0, \quad (15)$$

where,  $G_1 = r_1(T^*) - 2r_{10}N_1^*/K_{10} = -r_1(T^*)$ ,  $G_2 = -r_{20} + a_2N_1^*/(1 + B(T^* - T_0))$ ,  $G_3 = -1/\tau - \beta Z^*$ ,  $G_4 = -\beta C^*$ ,  $G_5 = -1/\tau_1$ ,  $G_6 = -\beta Z^*$ ,  $G_7 = -(\alpha_2 + \beta C^*)$ ,  $G_8 = -\alpha_1$

From the characteristic equation (15) we find that the equilibrium point  $E_2$  is linearly asymptotically stable under the condition given by:

$$\frac{r_{10}}{K_{10}} < \frac{\gamma a_1 r_{11} (T^* - T_0)}{\gamma a_1 K_{10} - r_{20} (1 + B(T^* - T_0))} \text{ and } T^* > T_0 \quad (16)$$

The characteristic equation associated with the variational matrix about equilibrium  $E_3$  is given by

$$(P_7 - \lambda)(\alpha_1 + \lambda)\{P_6 P_8 - (P_5 - \lambda)(P_9 - \lambda)\} \times \{(P_1 - \lambda)(P_4 - \lambda) - P_2 P_3\} = 0, \quad (17)$$

where,  $P_1 = r_1(T^*) - \frac{a_1 N_2^*}{1 + B(T^* - T_0)} - \frac{2r_{10}N_1^*}{K_{10}} = -\frac{r_{10}r_{20}}{K_{10}\gamma a_1} (1 + B(T^* - T_0))$ ,  $P_2 = -\frac{a_1 N_1^*}{1 + B(T^* - T_0)}$ ,  $P_3 = \frac{\gamma a_1 N_2^*}{1 + B(T^* - T_0)}$ ,  $P_4 = -r_{20} + \frac{\gamma a_1 N_1^*}{1 + B(T^* - T_0)}$ ,  $P_5 = -\frac{1}{\tau} - \beta Z^*$ ,  $P_6 = -\beta C^*$ ,  $P_7 = -1/\tau_1$ ,  $P_8 = -\beta Z^*$ ,  $P_9 = -(\alpha_2 + \beta C^*)$

From the nature of the roots of characteristic equation (17) we find that the equilibrium point  $E_3$  is linearly asymptotically stable provided  $T^* > T_0$ .

Further, from the above analysis it is noted that  $E_2$  is linearly stable only when  $E_3$  does not exist and  $E_3$  is linearly stable only if  $E_2$  is unstable.

It is observed from the analysis that the stability conditions are depending upon the equilibrium temperature level and average normal temperature.

### B. Global Stability

Next, we discuss the global stability of the interior equilibrium point  $E_3$ .

**Theorem 3.1:** The box  $V_1$  is a compact positively invariant set in space  $(N_1, N_2, C, C_1, Z, T)$ .

**Proof:** Consider the system given by Eqs. (2)-(7). To prove the theorem, we consider the point  $X' = (N_1', N_2', C', C_1', Z', T')$  out side the box  $V_1$ , with  $N_1' > K_{10}, N_2' > r_{10}K_{10}/\eta, C' > P\tau, C_1' > (I_0 + Q_0)/\eta, Z' > Q_0/\alpha_2$  and  $T' > D/\alpha_1$  and take the box  $V_1$  in the phase space  $(N_1, N_2, C, C_1, Z, T)$  with one vertex located at the origin and the other at  $X'$ . Now, let us compute the angle that the flow makes with each one of the faces of  $V_1$  not lying on the coordinate planes. Consider the planes  $\Pi_{N_1} : N_1 = N_1', \Pi_{N_2} : N_2 = N_2', \Pi_C : C = C', \Pi_{C_1} : C_1 = C_1', \Pi_Z : Z = Z'$  and  $\Pi_T : T = T'$  and let  $n_{N_1}, n_{N_2}, n_C, n_{C_1}, n_Z$  and  $n_T$  are outward unit normal vectors (with respect to box  $V_1$ ) respectively to each plane. Then

$$n_{N_1} \frac{dX}{dt} \Big|_{\pi_{N_1}} = N_1' \left( r_1(T') - \frac{r_{10}N_1'}{K_{10}} - \frac{a_1N_2'}{1 + B(T' - T_0)} \right)$$

then we get

$$n_{N_1} \frac{dX}{dt} \Big|_{\pi_{N_1}} \leq N_1' \left( -r_{11}T' - \frac{a_1N_2'}{1 + B(T' - T_0)} \right)$$

hence,

$$n_{N_1} \frac{dX}{dt} \Big|_{\pi_{N_1}} \leq 0$$

Similarly we can show that

$$n_{N_2} \frac{dX}{dt} \Big|_{\pi_{N_2}} \leq 0, n_C \frac{dX}{dt} \Big|_{\pi_C} \leq 0, n_{C_1} \frac{dX}{dt} \Big|_{\pi_{C_1}} \leq 0,$$

$$n_Z \frac{dX}{dt} \Big|_{\pi_Z} \leq 0, n_T \frac{dX}{dt} \Big|_{\pi_T} \leq 0$$

where,

$$\frac{dX}{dt} = (dN_1/dt, dN_2/dt, dC/dt, dC_1/dt, dZ/dt, dT/dt).$$

Thus, the flow along the normals to each of the plane is again moving towards the box. Clearly we can say that box  $V_1$  is compact positively invariant box. This completes the proof of the theorem 3.1. Now it is clear by the above theorem that the trajectories of the system cannot cross  $V_1$  once they enter inside. It is also observed that the interior equilibrium  $E_3$  lies inside  $V_1$ . Moreover,  $E_3$  is only attractor inside  $V_1$ , which is established in the following theorem.

**Theorem 3.2:** The equilibrium  $E_3$  is non-linearly

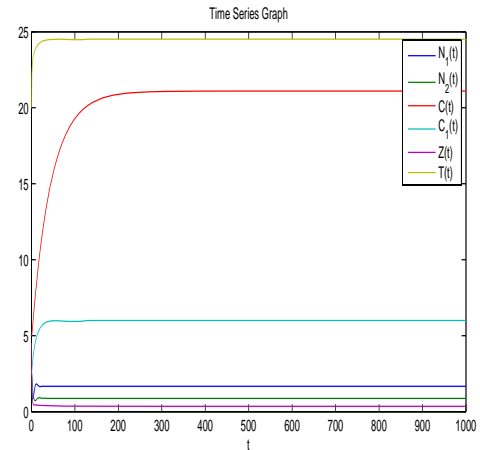


Fig. 4. Behavior of the trajectories of model with respect to time when  $B < B_0$ .

asymptotically stable with respect to solution initiating in the interior of  $V_1$  if the following inequalities hold:

$$A_1 r_{10} \left( \frac{\gamma a_1 N_M}{1 + B(T_m - T_0)} + r_{20} \right) > \frac{K_{10} a_1^2 (1 - \gamma N_2^* A_1)^2}{(1 + B(T_m - T_0))^2} \quad (18)$$

and

$$2A_4 A_2 (\alpha_2 + \beta C^*) \left( \frac{1}{\tau} + \beta Z_m \right) > \beta^2 (A_4 C^* + A_2 Z_m)^2 \quad (19)$$

(for proof see Appendix B)

### IV. NUMERICAL EXAMPLE

For the model, consider the following values of parameters-  $r_{10} = 0.9, r_{20} = 0.5, K_{10} = 3.0, r_{11} = 0.001, a_1 = 0.4, \gamma = 0.75, P = 0.5,$

$T_0 = 22, \tau = 60.0, \tau_1 = 10.0, \beta = 0.02, I_0 = 0.6, Q_0 = 0.5, C_{10} = 0.5,$

$h = 2.5, \alpha_1 = 1.2, \alpha_2 = 1.0, K_1 = 0.5, K_2 = 1.5.$

For the above set of values of parameters and  $B = 0.001 < B_0 = 0.287$ , we obtain the following value of interior equilibrium point  $E_3(N_1^*, N_2^*, C^*, C_1^*, Z^*, T^*)$

$N_1^* = 1.67, N_2^* = 0.87, C^* = 21.09, C_1^* = 6.0, Z^* = 0.35, T^* = 24.52$

It is noted that for the above set of parametric values, the stability conditions (13), (18) and (19) are satisfied. Hence,  $E_3^*$  is globally asymptotically stable (see Fig. 4).

For the above parametric values, when  $B = 0.288 > B_0 = 0.287$  is considered then the conditions for the existence of the interior equilibrium point  $E_3$  is violated and in this case the equilibrium point  $E_2$  exists with the following equilibrium values:

$N_1^* = 2.83, N_2^* = 0.00, C^* = 21.09, C_1^* = 6.0, Z^* = 0.35, T^* = 24.52$

The linear stability condition given by (16) for  $E_2$  is satisfied for the above set of parametric values with  $B = 0.0238$  and the stability behaviour of the model for  $B > B_0$  is shown in Fig. 5.

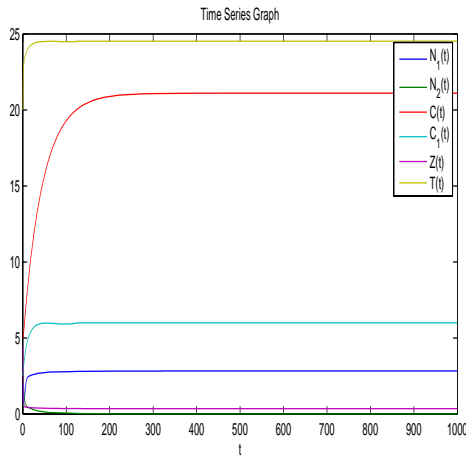


Fig. 5. Behavior of the trajectories of model with respect to time when  $B > B_0$ .

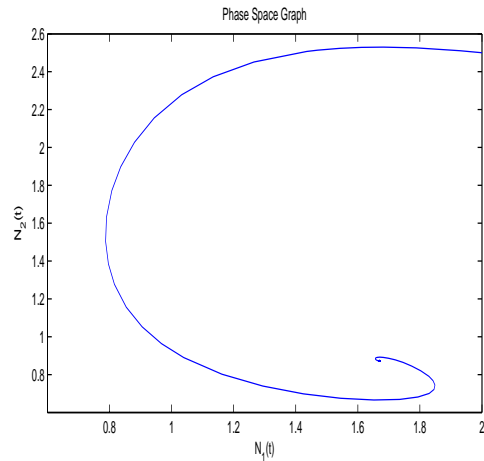


Fig. 6. Phase space diagram for prey population  $N_1(t)$  and predator population  $N_2(t)$  when  $B < B_0$ .

### V. CONCLUSION

From the linear stability analysis of the equilibrium point  $E_2$  it is concluded that the population with density  $N_2$  would tend to extinction and population with density  $N_1$  would survive but at lower equilibrium value due to the decrease in its growth rate on account of elevated temperature. The non-trivial positive equilibrium point  $E_3$  exists only when the equilibrium point  $E_2$  is unstable. Hence, from the linear as well as non-linear stability analysis of the non-trivial positive equilibrium  $E_3$  it is concluded that the prey and predator populations would co-exist if the parameter  $B$  which measures the stress of temperature is less than its threshold value  $B_0$  (see Figs. 4 and 6). However, it may be noted here that the prey and predator populations would co-exist even if the threshold value  $B_0$  is taken to be zero. Further, it is shown that if the value of the parameter  $B$  is more than its threshold value  $B_0$  then the predator population tend to extinction (see Figs. 5 and 7). Numerical example has been supplemented to validate the analytical results. The graphs of all the variables have been plotted with respect to time and from these graphs the stability behaviour is illustrated (see Figs. 4 and 5).

#### APPENDIX A PROOF OF LEMMA 3.1

**Proof:** From Eqs. (2) and (3) we get,

$$\begin{aligned} \frac{d(N_1 + N_2)}{dt} &\leq (r_{10} - r_{11}(T - T_0))N_1 - r_{20}N_2 \\ &\quad - \frac{N_1N_2}{1 + B(T - T_0)}(a_1 - a_2) \\ &\leq r_{10}K_{10} - r_{11}(T_m - T_0)N_1 - r_{20}N_2 - \frac{N_1N_2}{1 + B(T - T_0)}(a_1 - a_2) \\ &\leq r_{10}K_{10} - \eta(N_1 + N_2) \end{aligned}$$

if  $a_2 < a_1$ , i.e.  $\gamma < 1$ .

Where  $\eta = \min(r_{11}(T_m - T_0), r_{20})$  and it is assumed that  $T_m > T_0$ .

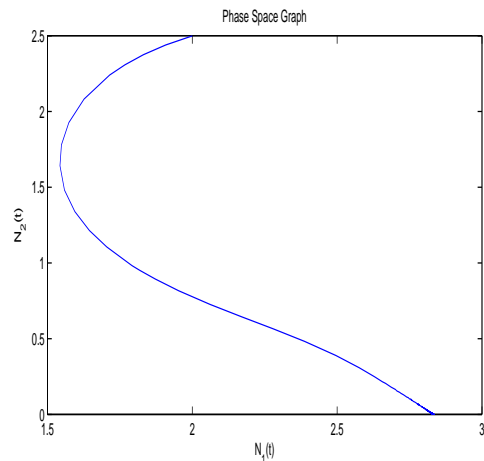


Fig. 7. Phase space diagram for prey population  $N_1(t)$  and predator population  $N_2(t)$  when  $B > B_0$ .

Then by the usual comparison theorem we get as  $t \rightarrow \infty$  :

$$(N_1 + N_2) \leq \frac{r_{10}K_{10}}{\eta}$$

and hence,

$$N_2 \leq \frac{r_{10}K_{10}}{\eta}$$

From Eq. (7) and using the solution of  $C_1$  in eq.(7) we get,

$$\frac{dT}{dt} \leq D + hBe^{-t/\tau_1} - \alpha_1 T$$

where,  $A = \frac{K_1}{K_2 + Z_m} + \alpha_1 T_0$  and  $D = hI_0\tau_1 + A$

Then by the usual comparison theorem we get as  $t \rightarrow \infty$  :

$$T \leq \frac{D}{\alpha_1}$$

i.e.

$$T \leq \frac{hI_0\tau_1}{\alpha_1} + \frac{K_1}{\alpha_1(K_2 + Z_m)} + T_0 = T_M$$

Again from Eq. (7) we get ,

$$\frac{dT}{dt} \geq \left( \frac{K_1}{K_2 + Z_M} - hC_{10} \right) - \alpha_1 T$$

Then by the usual comparison theorem we get as  $t \rightarrow \infty$  :

$$T \geq \frac{\frac{K_1}{K_2 + Z_M} - hC_{10}}{\alpha_1} = T_m$$

and  $T_m > 0$  if  $\frac{K_1}{K_2 + Z_M} > hC_{10}$

Similarly from Eqs. (4) and (6), we get as  $t \rightarrow \infty$  :

$$C \leq P\tau = C_M, \quad Z \leq \frac{Q_0}{\alpha_2} = Z_M$$

Again from Eq. (6) we get

$$\frac{dZ}{dt} \geq Q_0 - \alpha_2 Z - \beta Z P\tau$$

By the usual comparison theorem we get as  $t \rightarrow \infty$  :

$$Z \geq \frac{Q_0}{\alpha_2 + \beta P\tau} = Z_m$$

By adding Eqs. (5) and (6), we get

$$\frac{d(C_1 + Z)}{dt} \leq I_0 + Q_0 - \eta(C_1 + Z)$$

where,  $\eta = \min(\frac{1}{\tau_1}, \alpha_2)$  and by usual comparison theorem we get as  $t \rightarrow \infty$  :

$$(C_1 + Z) \leq \frac{I_0 + Q_0}{\eta}$$

and hence,

$$C_1 \leq \frac{I_0 + Q_0}{\eta} = C_{1M}$$

This completes the proof of the lemma 3.1.

#### APPENDIX B PROOF OF THEOREM 3.2

**Proof:** Taking the perturbations about the equilibrium value as follows:

$$N_1 = N_1^* + u_1(t), \quad N_2 = N_2^* + u_2(t), \quad C = C^* + v_1(t),$$

$$C_1 = C_1^* + w_1(t), \quad Z = Z^* + x(t), \quad T = T^* + t_1(t).$$

the non-linearised system of equations from (2) to (7) about equilibrium point  $E_3$  is given by

$$\frac{du_1}{dt} = (N_1^* + u_1) \left\{ -r_{11}t_1 - \frac{a_1 N_2^* B t_1}{(1 + B(T^* - T_0))(1 + B(T - T_0))} - \frac{r_{10}u_1}{K_{10}} - \frac{a_1 u_2}{1 + B(T - T_0)} \right\} \quad (20)$$

$$\frac{du_2}{dt} = -r_{20}u_2 - \frac{B t_1 \gamma a_1 N_1^* N_2^*}{(1 + B(T - T_0))(1 + B(T^* - T_0))} + \frac{\gamma a_1}{1 + B(T - T_0)} (N_1 u_2 + N_2^* u_1) \quad (21)$$

$$\frac{dv_1}{dt} = -\frac{v_1}{\tau} - \beta C^* x - \beta(Z^* + x)v_1 \quad (22)$$

$$\frac{dw_1}{dt} = -\frac{w_1}{\tau_1} \quad (23)$$

$$\frac{dx}{dt} = -\alpha_2 x - \beta C^* x - \beta(Z^* + x)v_1 \quad (24)$$

$$\frac{dt_1}{dt} = \frac{-K_1}{(K_2 + Z^*)(K_2 + Z^* + x)} + h w_1 - \alpha_1 t_1 \quad (25)$$

Consider,

$$G(t) = \left[ u_1 - N_1^* \log\left(1 + \frac{u_1}{N_1^*}\right) \right] + \frac{1}{2} A_1 u_2^2 + \frac{1}{2} A_2 v_1^2 + \frac{1}{2} A_3 w_1^2 + \frac{1}{2} A_4 x^2 + \frac{1}{2} A_5 t_1^2$$

where,  $A_i (i = 1$  to  $5)$  are arbitrary positive constants.

The time derivative of  $G(t)$  is given by

$$\frac{dG}{dt} = \frac{u_1}{N_1^* + u_1} \frac{du_1}{dt} + A_1 u_2 \frac{du_2}{dt} + A_2 v_1 \frac{dv_1}{dt} + A_3 w_1 \frac{dw_1}{dt} + A_4 x_1 \frac{dx_1}{dt} + A_5 t_1 \frac{dt_1}{dt}$$

Now, using the system of equations (20)-(25) in  $dG/dt$  in the region  $V_1$ , we get

$$\begin{aligned} \frac{dG}{dt} \leq & -\left\{ \frac{1}{2} \frac{r_{10}}{K_{10}} u_1^2 + \frac{a_1}{1 + B(T_m - T_0)} (1 - \gamma N_2^* A_1) u_1 u_2 \right. \\ & + \frac{A_1}{2} \left( \frac{\gamma a_1 N_M}{1 + B(T_m - T_0)} + r_{20} \right) u_2^2 + \frac{1}{2} \frac{r_{10}}{K_{10}} u_1^2 \\ & + \left( r_{11} + \frac{a_1 N_2^* B}{(1 + B(T^* - T_0))(1 + B(T_m - T_0))} \right) u_1 t_1 \\ & + \frac{A_5}{4} \alpha_1 t_1^2 + \frac{A_1}{2} \left( \frac{\gamma a_1 N_M}{1 + B(T_m - T_0)} + r_{20} \right) u_2^2 \\ & + \frac{A_1 B \gamma a_1 N_1^* N_2^*}{(1 + B(T^* - T_0))(1 + B(T_m - T_0))} u_2 t_1 + \frac{A_5}{4} \alpha_1 t_1^2 \\ & + \frac{A_4}{2} (\alpha_2 + \beta C^*) x^2 + \beta (A_4 C^* + A_2 Z_m) x v_1 \\ & + A_2 \left( \frac{1}{\tau} + \beta Z_m \right) v_1^2 + \frac{A_4}{2} (\alpha_2 + \beta C^*) x^2 \\ & + \frac{A_5 K_1}{(K_2 + Z^*)(K_2 + Z_M)} x t_1 + \frac{A_5}{4} \alpha_1 t_1^2 \\ & \left. + \frac{A_3}{\tau} w_1^2 - A_5 h w_1 t_1 + \frac{A_5}{4} \alpha_1 t_1^2 \right\} \end{aligned}$$

Using the sylvester's criteria in the right hand side of the above expression and then choosing  $A_1, A_3, A_4, A_5$  as follow

$$A_5 > 2 \frac{K_{10}}{\alpha_1 r_{10}} \left[ r_{11} + \frac{a_1 N_2^* B}{(1 + B(T^* - T_0))(1 + B(T_m - T_0))} \right]^2, \\ A_1 \left[ \frac{B \gamma a_1 N_1^* N_2^*}{(1 + B(T^* - T_0))(1 + B(T_m - T_0))} \right]^2 < \frac{A_5}{2} \left[ r_{20} + \frac{\gamma a_1 N_M}{1 + B(T_m - T_0)} \right] \alpha_1$$

and

$$A_4(\alpha_2 + \beta C^*) > 2A_5 \left[ \frac{K_1}{(K_2 + Z)(K_2 + Z_m)} \right]^2$$

it may be shown that  $dG/dt$  is negative definite if the conditions (18) and (19) are satisfied. Thus, it is proved that  $E_3$  is globally (non-linearly) asymptotically stable in the region  $V_1$ .

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